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Generally Covariant Schrödinger Equation in Newton–Cartan Space–Time. Part II

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Schrödinger equation in Newton–Cartan space–time can be obtained from Einstein equivalence principle, that is firstly one should obtain generally covariant Schrödinger's equation in Galilean space–time (using generally covariant Hamilton–Jacobi formalism and Schrödinger's Ansatz, as was previously shown) and get Schrödinger's equation in Newton–Cartan space–time with the help of equivalence principle. The equation possesses a gauge freedom f connected with phase transformation of the wave function. But absolute elements of the space–time possess a symmetry group and they depend on f. So, a natural problem arises: to find the gauge f in which absolute elements become *invariant* with respect to the group. In the paper the gauge f is found with the help of space–time properties, that is transformation rule of the wave function is obtained from the space–time structure (in Newton–Cartan space–time). The special form of f is found in the case when the space–time as a whole possesses a symmetry.

1. INTRODUCTION

Quantum mechanics in the gravity field is one of the most important problem in physics. Moreover, there has been great interest in the quantum mechanics in Newton–Cartan space–time of late; see for example Penrose (1997), Lämmerzahl (1996), and Kasevich and Chu (1991).

The paper is an direct continuation of Wawrzycki (2001) where the generally covariant Schrödinger equation was obtained in Newton–Cartan space–time. The equation possesses a gauge freedom f that transforms the phase of the wave function. In the case of the flat space–time this additional degree of freedom was uniquely eliminated by space–time symmetries. This means that transformation properties of the wave function follow from space–time properties, see Wawrzycki (2001).

It is shown in this paper how the phase f is established by space–time properties when gravity field is present. Newton–Cartan space–time as a whole does

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not possess any symmetry (in general case) and one would expect the problem to be much more difficult than in flat space–time. Moreover, one could suppose that the problem in curved space–time is not well defined. But the Newton–Cartan space–time possesses absolute elements in addition to dynamical ones (contrary to Theory of General Relativity). The absolute elements have a symmetry group. They enter the Schrödinger equation and depend on f, so a natural problem arises: to find explicit form of f, which brings the absolute elements into the form *invariant* with respect to the symmetry group. This is enough to establish f in the manner analogous to that used in the Galilean space–time.

In other words the Schrödinger equation follows from the equivalence principle and its form in the privileged coordinates is

$$i\hbar\, \partial_t \Psi = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \Psi + m\varphi\Psi \tag{1}$$

with the inertial mass equal to the gravitational one, where φ denotes Newtonian potential fulfilling Laplace's equation.

The paper is organized as follows. In section 2 absolute and dynamical objects of Newton–Cartan space–time are described. In section 3 gauge freedom is eliminated by symmetry of absolute objects. In section 4 space–time with an additional symmetry is considered. In section 5 Schrödinger equation in privileged coordinate system is obtained.

2. ABSOLUTE AND DYNAMICAL OBJECTS OF NEWTON–CARTAN SPACE–TIME

Three independent objects describe Newton–Cartan space–time: the connection $\Gamma^{\sigma}_{\mu\nu}$, the gradient of absolute time t_μ, and contravariant tensor field $g^{\mu\nu}$.

They fulfil the following postulates (after Trautman, 1963):

I. Affine connection is torsion-free, $\Gamma^{\mu}_{[\nu\rho]} = 0$.

II. $R^{\mu}_{\mu\rho\sigma} = 0$.

III. There exist three linearly independent vector fields ξ_i^{μ}, such that

$$\xi_i^{\mu} t_\mu = 0, \qquad \nabla_\nu \xi_i^{\mu} = 0,$$

where $i = 1, 2, 3$. This is equivalent to

$$t_{[\lambda} R^{\mu}_{\nu]\rho\sigma} = 0.$$

IV. Curvature tensor is such that

$$R^{\mu\lambda}_{\nu\sigma} = R^{\lambda\mu}_{\sigma\nu}, \quad \text{where} \quad R^{\mu\lambda}_{\nu\sigma} = g^{\lambda\rho} R^{\mu}_{\nu\rho\sigma}.$$

V. $\nabla_\lambda g_{\mu\nu} = 0$.

VI. The rank of $g^{\mu\nu}$ is equal to three and

$$g^{\mu\nu} t_\mu = 0.$$

VII. The space–time curves of a free-falling particles are geodesics of $\Gamma^\mu_{\nu\rho}$.

VIII. Equations of the gravity field in the vacuum are

$$R_{\mu\nu} = 0,$$

where $R_{\mu\nu} = R^\lambda_{\mu\nu\lambda}$ is the Ricci tensor.

It can be shown that

$$\nabla_\nu t_\mu = 0 \tag{2}$$

for the gradient of the absolute time t.

From I, II, and III it follows that (Trautman, 1963)

$$R^\mu_{\nu\rho\sigma} = 2 t_\nu \varphi^\mu{}_{[\rho} t_{\sigma]},$$

where

$$\varphi^\mu_\rho = \nabla_\rho \varphi^\mu \quad \text{and} \quad \varphi^\mu t_\mu = 0. \tag{3}$$

The following substitutions do not change the value of the curvature tensor

$$\varphi^\mu \to \varphi^\mu + a^i \xi^\mu_i, \tag{4}$$

where $a^i = a^i(t)$ are three arbitrary functions of time. Connection can be written in the following form

$$\Gamma^\mu_{\nu\rho} = \mathring{\Gamma}^\mu_{\nu\rho} + \varphi^\mu t_\nu t_\rho, \tag{5}$$

where $\mathring{\Gamma}^\mu_{\nu\rho}$ is an integrable connection, that is $\mathring{R}^\mu_{\nu\rho\sigma} = 0$. From IV follows that

$$\varphi^\mu = g^{\mu\nu}\, \partial_\nu \varphi.$$

Field equations VIII have the form

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi = 0 \quad \text{or} \quad g^{\mu\nu}\, \mathring{\nabla}_\mu\, \mathring{\nabla}_\nu \varphi = 0, \tag{6}$$

where $\mathring{\nabla}_\mu$ is the covariant derivative with respect to $\mathring{\Gamma}^\mu_{\nu\rho}$. From (6) follows that φ is determined up to an arbitrary function of time. Because of (3) and (5), V and (2) can be written in the form

$$\mathring{\nabla}_\mu t_\nu = 0 \quad \text{and} \quad \mathring{\nabla}_\mu g^{\nu\rho} = 0. \tag{7}$$

So, φ is the only dynamical object. Among the quantities t_μ, $g^{\mu\nu}$, and $\Gamma^\mu_{\nu\rho}$ only the connection is dynamical. With the help of the connection, however, covariant tensor $g_{\mu\nu}$ and contravariant vector $\boldsymbol{u}^\mu$ (first introduced by Daŭtcourt, 1990, see also Wawrzycki, 2001) can be defined. $g_{\mu\nu}$ fulfils

$$g^{\lambda\nu} g_{\nu\beta} g^{\beta\mu} = g^{\lambda\mu}.$$

Lagrange function of a free-falling particle (motion of which is described by a curve $x^{\mu}(\tau)$, where τ is an parameter along the curve not necessarily equal to the absolute time) is

$$L = \frac{m}{2}\frac{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{t_{\sigma}\dot{x}^{\sigma}}, \tag{8}$$

where the full-parameter derivative is marked by dot. Lagrange function is determined up to full-parameter derivative and $g_{\mu\nu}$ is determined up to gauge transformation

$$g_{\mu\nu} \to g_{\mu\nu} + t_{\mu}\,\partial_{\nu} f + t_{\nu}\partial_{\mu} f. \tag{9}$$

$\boldsymbol{u}^{\mu}$ is defined in the following way

$$\begin{aligned} g_{\mu\nu}g^{\nu\sigma} &= \delta^{\sigma}_{\mu} - \boldsymbol{u}^{\sigma}t_{\mu}, \\ \boldsymbol{u}^{\mu}t_{\mu} &= 1, \end{aligned} \tag{10}$$

up to the gauge transformation

$$\boldsymbol{u}^{\mu} \to \boldsymbol{u}^{\mu} - g^{\mu\nu}\,\partial_{\nu} f. \tag{11}$$

Lagrange–Euler equations of the Lagrange function (8) give

$$\Gamma^{\mu}_{\nu\rho} = \boldsymbol{u}^{\mu}\partial_{\nu}t_{\rho} + \frac{1}{2}g^{\mu\sigma}\{\partial_{\nu}g_{\rho\sigma} + \partial_{\rho}g_{\nu\sigma} - \partial_{\sigma}g_{\nu\rho}\}. \tag{12}$$

Conversely, (10) and (12) up to the gauge determine covariant g and contravariant $\boldsymbol{u}$, so, they completely replace the affine connection. Formula (12) is gauge independent. Instead of I, II, III, and IV covariant g and contravariant $\boldsymbol{u}$ can be introduced with the help of the formula (10) and (12).

From I, II, and III it follows that there exists a *gauge* in which covariant g can be decomposed into two parts

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} - 2\varphi t_{\mu}t_{\nu}, \tag{13}$$

where $\Phi = g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu} = -2\varphi$. From this decomposition follows that $\mathring{g}_{\mu\nu}\boldsymbol{u}^{\mu} = 0$. After direct calculation one gets $\mathring{R}^{\mu}_{\nu\rho\sigma} = 0$ and the connection

$$\mathring{\Gamma}^{\mu}_{\nu\rho} = \boldsymbol{u}^{\mu}\,\partial_{\nu}t_{\rho} + \frac{1}{2}g^{\mu\sigma}\{\partial_{\nu}\mathring{g}_{\rho\sigma} + \partial_{\rho}\mathring{g}_{\nu\sigma} - \partial_{\sigma}\mathring{g}_{t\rho}\}$$

is integrable and $\mathring{\nabla}_{\rho}\mathring{g}_{\mu\nu} = 0$. Contravariant $\boldsymbol{u}$ defined by (10) does not depend on φ and is an absolute object of space–time as well as $\mathring{g}_{\mu\nu}$. In section 3 the *gauge* mentioned earlier will be found (the explicit form of the phase f connected with a coordinate transformation) with the help of symmetry of absolute objects (axioms I, II, III, and IV will not be used).

3. TRANSFORMATION RULE OF THE WAVE FUNCTION

Generally covariant Schrödinger equation in Newton–Cartan space–time has the form (Wawrzycki, 2001)

$$i\hbar \boldsymbol{u}^{\mu}\,\partial_{\mu}\Psi = \frac{\hbar^2}{2m}g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\Psi - \frac{m}{2}g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}\Psi - \frac{i\hbar}{2}\nabla_{\mu}\boldsymbol{u}^{\mu}\Psi. \tag{14}$$

The equation may be found with the help of Schrödinger's Ansatz using the covariant Hamilton–Jacobi equation, which has the form (see Wawrzycki, 2001)

$$\boldsymbol{u}^{\mu}\,\partial_{\mu}S + \frac{1}{2m}g^{\mu\nu}\,\partial_{\mu}S\,\partial_{\nu}S - \frac{m}{2}g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu} = 0. \tag{15}$$

Equation (14) follows from the variational principle

$$\delta\left(\int \Lambda v\, d^4x\right) = 0,$$

where v is the *invariant measure* (scalar density of weight $+1$) of Newton–Cartan space–time, which is equal

$$v = \sqrt{\det(g_{\mu\nu} + (1 - g_{\alpha\beta}\boldsymbol{u}^{\alpha}\boldsymbol{u}^{\beta})t_{\mu}t_{\nu})} \tag{16}$$

and Λ is a scalar

$$\Lambda = \frac{i\hbar}{2}\boldsymbol{u}^{\mu}(\Psi^*\nabla_{\mu}\Psi - \Psi\nabla_{\mu}\Psi^*) - \frac{\hbar^2}{2m}g^{\mu\nu}\nabla_{\mu}\Psi\nabla_{\nu}\Psi^* + \frac{m}{2}g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}\Psi\Psi^*, \tag{17}$$

such that Λv is the Lagrange density of the Schrödinger equation.

Equations (14) and (15) are covariant with respect to coordinate transformations and gauge transformations (9) and (11), where the gauge transforms S and Ψ in the following way

$$S \to S + mf \quad \text{and} \quad \Psi \to e^{\frac{im}{\hbar}f}\Psi.$$

(16) and (17) are gauge independent of course.

One can combine gauge transformation and coordinate transformation. In the case of the flat space–time there exists a combination that brings $g_{\mu\nu}$ and $\boldsymbol{u}^{\mu}$ into the form *invariant* with respect to Galilean transformations (see Wawrzycki, 2001). This establishes f. Namely, they have the transformation rule

$$\begin{aligned} g_{\mu\nu} &\to \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}(g_{\mu\nu} + t_{\mu}\partial_{\nu}f + t_{\nu}\partial_{\mu}f) \\ \boldsymbol{u}^{\mu} &\to \frac{\partial x^{\mu'}}{\partial x^{\mu}}(\boldsymbol{u}^{\mu} - g^{\mu\nu}\,\partial_{\nu}f) \end{aligned} \tag{18}$$

where the coordinate transformation $x^\mu \to x^{\mu'}$ (this will be denoted by $(\mu) \to (\mu'))$ is combined with appropriate gauge transformation f.

Consider now the point transformation (not coordinate transformation) connected with the above coordinate transformation in such a way that *coordinate system* (μ) *is dragged along of the coordinate system* (μ') by the point transformation, see Schouten (1951). That is, each point has the same coordinates with respect to (μ) as its image (by the point transformation) has with respect to (μ').

Now, let some field S (eventual indices are omitted) be given. Let $\hat{S}$ be a second field, whose components with respect to (μ') are equal to the components of the first field S in the corresponding point (with respect to (μ)). If then $S = \hat{S}$, the field S is called *invariant for the point transformation*.

Substituting $g_{\mu\nu}$ and $\boldsymbol{u}^\mu$ for S with their transformation laws (18) one gets

$$
\begin{aligned}
-\overline{\delta}_{\delta x^\lambda} g_{\mu\nu} &= \partial_\mu(\delta x^\rho) g_{\rho\nu} + \partial_\nu(\delta x^\rho) g_{\mu\rho} + \delta x^\rho \partial_\rho g_{\mu\nu} \\
&= t_\mu \partial_\nu f(\delta x) + t_\nu \partial_\mu f(\delta x) \qquad (19) \\
\overline{\delta}_{\delta x^\lambda} \boldsymbol{u}^\mu &= \partial_\nu(\delta x^\mu)\boldsymbol{u}^\nu - \delta x^\nu \partial_\nu \boldsymbol{u}^\mu = g^{\mu\nu}\partial_\nu f(\delta x),
\end{aligned}
$$

where infinitesimal point transformation $x^\mu \to x^\mu + \delta x^\mu$ was substituted. So, (19) are *invariance conditions* of covariant g and contravariant $\boldsymbol{u}$ for a point transformation δx^μ.

In the case of the flat space–time (19) are fulfilled if and only if δx is a space–time symmetry (an element of the inhomogeneous Galilei group) and f is equal to *galilean phase* (see Wawrzycki, 2001).

Consider now the Newton–Cartan space–time. One has the following theorems

Theorem 1. *The rigid nonrotating motions* $\delta x^\mu = dR^i \xi_i^\mu = \dot{R}^i \xi_i^\mu dt$ *(see section* 2 *postulate* III *for definition of* ξ_i^μ, *where* R^i *are arbitrary functions of time, dot denotes time derivative) and time translations compose symmetry group of absolute elements, i.e. rigid nonrotating motions* δx *are the only transformations, which fulfil*

A. $\underset{\delta x}{\overline{\delta}}\, g^{\mu\nu} = 0.$

B. $\underset{\delta x}{\overline{\delta}}\, t_\mu = 0.$

C. $\underset{\delta x}{\overline{\delta}}\, \boldsymbol{u}^\mu = g^{\mu\nu}\, \partial_\nu f(\delta x)$. *(Obviously all rigid time-dependent motions* δx *fulfil* A *and* B. C *eliminates time-dependent rotations. From* A, B, C, *and (10) one gets D (follows from (19), but not conversely).)*

D. $-\underset{\delta x}{\overline{\delta}}\, g_{\mu\nu} = t_\mu\, \partial_\nu f(\delta x) + t_\nu\, \partial_\mu f(\delta x) - \underset{\delta x}{\overline{\delta}}\, g_{\alpha\beta}\boldsymbol{u}^\alpha \boldsymbol{u}^\beta t_\mu t_\nu - 2u^\alpha\, \partial_\alpha f(\delta x) t_\mu t_\nu.$

Theorem 2. C *and* D *determinates*

a. $\boldsymbol{u}^{\mu}$ *uniquely,*
b. *the phase* f *up to an arbitrary function of time* $\bar{f}(t)$*:*

$$f \to f' = f + \bar{f},$$

c. $g_{\mu\nu}$ *up to an arbitrary function of time* $\bar{\bar{f}}(t)$*:*

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + t_{\mu}\,\partial_{\nu}\,\bar{\bar{f}} + t_{\nu}\,\partial_{\mu}\,\bar{\bar{f}} = g_{\mu\nu} + 2\,\dot{\bar{\bar{f}}}(t) t_{\mu} t_{\nu},$$

where dot denotes time derivative.

One can check it directly by substituting $g'_{\mu\nu}$, f', and $\boldsymbol{u}'^{\mu}$ to C and D (of course $\boldsymbol{u}^{\mu} \to \boldsymbol{u}'^{\mu} = \boldsymbol{u}^{\mu} - g^{\mu\nu}\,\partial_{\nu}\,\bar{\bar{f}}$).

Theorem 3. *Solution of* C *and* D *is as follows*

a. $g_{\mu\nu} = \mathring{g}_{\mu\nu} - 2\varphi t_{\mu} t_{\nu}$ *mentioned in section* 2*, where*

$$2\varphi = -g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu} \equiv -\eta, \quad \mathring{\nabla}_{\rho}\mathring{g}_{\mu\nu} = 0, \quad \mathring{g}_{\mu\nu}\boldsymbol{u}^{\mu} = 0, \quad \textit{and}$$
$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\varphi = 0;$$

b. $\boldsymbol{u}^{\mu}$ *is such that* $\nabla_{\mu}\boldsymbol{u}^{\nu} = g^{\nu\rho}\,\partial_{\rho}\varphi t_{\mu}$;
c. gradient of the phase of the transformation δx *is equal*

$$\partial_{\mu} f[p] = -g_{\mu\lambda}\boldsymbol{v}^{\lambda} + \frac{1}{2} g_{\alpha\lambda}\boldsymbol{v}^{\alpha}\boldsymbol{v}^{\lambda} t_{\mu} - t_{\mu} \times \int_{C[p_{\circ},p]} \boldsymbol{u}^{\nu}\nabla_{\nu}\boldsymbol{v}^{\alpha} g_{\alpha\lambda}\, dx^{\lambda} + \bar{f}(t) t_{\mu}, \qquad (20)$$

where $\delta x^{\mu} = \dot{R}^{i}(t)\xi_{i}^{\mu}\, dt = \boldsymbol{v}^{\mu}\, dt$*, so,* $\boldsymbol{v}^{\mu}$ *is the tangent vector field of the integral curves of point transformation* δx^{μ} *understood as a one-parameter congruence (one-parameter group) with time* t *as the parameter,* $C[p_{\circ}, p]$ *is a contour with end-points in* $p_{\circ}$ *and* p*.*

The integral in the formula for gradient of f determinates some function of end-point p up to a function of time depending on the choice of the contour C. But f is determined only up to any function of time and there is no problem which would be connected with the multivaluedness property of the integral. One can check the result directly. Calculations are long but essentially simple.

Because of **Theorem 2** there do not exist any other solutions than that presented in **Theorem 3**. Equation (20) gives the phase of the *rigid non-rotating time-dependent transformation.* But because C and D define all possible $g_{\mu\nu}$ and

$\boldsymbol{u}^{\mu}$s, which differ from each other by a gauge transformations f determined by (20), (20) defines phase of any transformation $(x^{\nu}, \boldsymbol{u}^{\mu}) \to (x^{\nu'}, \boldsymbol{u}'^{\mu'})$ with $\boldsymbol{v}^{\mu} = \boldsymbol{u}^{\mu} - \boldsymbol{u}'^{\mu}$.

From C and D follows that $\mathring{g}_{\mu\nu}$ is *invariant* (in the geometrical sense mentioned earlier) for any rigid time-dependent transformation δx. We shall find invariance condition of that object. Firstly, one should find its gauge transformation, then combining it with coordinate transformation (like in (18)) one gets transformation rule

$$\mathring{g}_{\mu\nu} \to \mathring{g}'_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} (\mathring{g}_{\mu\nu} + t_{\mu}\,\partial_{\nu} f + t_{\nu}\,\partial_{\mu} f - \{2u^{\alpha}\,\partial_{\alpha} f - g^{\alpha\beta}\,\partial_{\alpha} f \partial_{\beta} f\} t_{\mu} t_{\nu}) \tag{18$'$}$$

Inserting to invariance definition (with the transformation rule in mind) one gets invariance condition

$$\mathrm{D}'. \; -\underset{\delta x}{\bar{\delta}}\, \mathring{g}_{\mu\nu} = t_{\mu}\,\partial_{\nu} f(\delta x) + t_{\nu}\,\partial_{\mu} f(\delta x) - 2u^{\alpha}\,\partial_{\alpha} f(\delta x) t_{\mu} t_{\nu}.$$

D$'$ follows from C and D as has been remarked earlier (in consequence from A, B, and C). Indeed, from the fact that

$$\underset{\delta x}{\bar{\delta}}(-2\varphi) = \underset{\delta x}{\bar{\delta}}\,\eta = \underset{\delta x}{\bar{\delta}}(g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}) = \underset{\delta x}{\bar{\delta}}\, g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}$$

and from D one gets D$'$. In the analogous way one can obtain invariance condition of the potential φ, or equivalently η,

$$\mathrm{E.} \quad \underset{\delta x}{\bar{\delta}}(\eta) = -2\boldsymbol{u}^{\mu}\,\partial_{\mu} f(\delta x)$$

but in this case E *is completely independent of* A, B, *and* C (which is obvious, because the Newton–Cartan space–time is not flat in general). Condition E is fulfilled if and only if space–time is flat. This is an immediate consequence of equivalence of invariance condition (19) for covariant g and condition E this equivalence can be easily seen if one takes into account D (see also discussion in section 4).

C and D are generally covariant with respect to (18) with f determined by (20) as well as D' is with respect to (18$'$). Moreover contravariant $\boldsymbol{u}$ and covariant $\mathring{g}$ are invariant with respect to time translations, space translations (this is evident, because space translation is a special case of rigid motion), and rotations. All invariance conditions are of the form C and D, and all of them are generally covariant. A comment should be given concerning this fact. The phase $f(\delta x)$ of a transformation δx depends on its velocity υ and acceleration. When the velocity of the considered transformation δx is equal to zero then its phase $f(\delta x)$ is equal to zero. On the other hand for any realization of *static symmetry* (translation or rotation) one can always choose a frame of reference in which velocity and phase of the transformation is zero. Invariance conditions for *static symmetry* are simplified in this frame and their right hand sides are equal to zero (and do not depend on f explicitly). But the same *symmetry* has *nonzero velocity in other frames of reference.* So, the terms on the right hand sides of invariance conditions depending on f *cannot be omitted even when the symmetry is static.*

As an example consider rotation invariance condition C of $\boldsymbol{u}$ in Galilean space–time in inertial reference frames (μ) and (μ') connected by a Galilean transformation $(\mu)\to(\mu')$. Infinitesimal rotation δx in (μ) and (μ') is of the form

$$(\delta x^{\mu}) = \begin{pmatrix} 0 \\ \varepsilon^{j}{}_{i} x^{i} \end{pmatrix} \quad \text{and} \quad (\delta x^{\mu'}) = \begin{pmatrix} 0 \\ \varepsilon^{j'}{}_{i'} x^{i'} - \varepsilon^{j'}{}_{i'} V^{i'} t \end{pmatrix}$$

respectively, where V^i is the velocity of the transformation $(\mu)\to(\mu')$ and $\varepsilon_{ij} = -\varepsilon_{ji}$. So, the velocity υ of the rotation δx in (μ) and (μ') is equal

$$\boldsymbol{\upsilon}^{\mu} = 0 \quad \text{and} \quad (\boldsymbol{\upsilon}^{\mu'}) = \begin{pmatrix} 0 \\ -\varepsilon^{j'}{}_{i'} V^{i'} \end{pmatrix}$$

respectively, and $f = 0$ in (μ) and $f \neq 0$ in (μ'), and C is fulfilled in (μ) and (μ').

4. SPACE–TIMES WITH A SYMMETRY ξ^{μ}

It would seem that the meaning of the words *symmetry of* φ *with respect to the transformation determined by integral curves of* ξ^{μ} and *constancy of the potential* φ *along those curves* is the same. But this is not exactly the same. The reason lies in the fact that potential is not a scalar (or equivalently, covariant *g* with their transformation rule (18) does not compose any tensorial quantity). Even for a tensor density the meaning is not the same.

If *the potential* φ *is constant along a family of integral curves of a given vector field* ξ^{μ}*, then*

$$\underset{\xi}{\bar{\delta}}\varphi = 0 \quad \text{or equivalently} \quad \underset{\xi}{\bar{\delta}}\eta = 0. \tag{21}$$

On the other hand invariance condition of the potential (or equivalently η) has the form E *not equivalent* to (21).

Let us suppose that space–time possesses a symmetry ξ (any symmetry *static* or not). Consider infinitesimal symmetry transformation $\delta\xi$ that can be decomposed into static part (with the subscript zero) and *dynamic* (with nonzero velocity)

$$\delta\xi^{\mu} = \xi^{\mu}_{\circ}\, d\tau + \boldsymbol{\upsilon}^{\mu}\, d\boldsymbol{\upsilon},$$

where $d\tau$ is the parameter of *static* transformation and $d\boldsymbol{\upsilon}$ is the parameter of dynamic transformation. The parameters are independent and even when the transformation $\delta\xi$ is static *dynamical* part cannot be eliminated, i.e. there always exists reference frame in which *dynamical* part is not equal to zero (see discussion at the end of preceding paragraph). A rigid non-rotating time-dependent transformation can be considered as a *dynamical* part.

Because η (equivalently potential φ) is invariant, for $\delta\xi$ the field dragged along of the field η is equal to η. The component η'_{dragg} of the field dragged along at $x + \delta\xi$ with respect to (μ') is equal to the component η in (μ) at x

$$\eta'_{\text{dragg}}(x^\mu + \delta\xi^\mu) = \eta(x^\mu),$$

which is equivalent to

$$\eta'_{\text{dragg}}(x^\mu) = \eta(x^\mu - \delta\xi^\mu),$$

where (μ') is the coordinate system dragged along by the point transformation $\delta\xi$. One must compute component of dragged field in (μ) at x and then compare it with the component of initial field in the same frame (μ) at the same point x. Because of the transformation rule

$$\eta' = \eta + 2\boldsymbol{u}^\mu\,\partial_\mu f(\boldsymbol{\upsilon}^\alpha\,d\boldsymbol{\upsilon}),$$

one has

$$\begin{aligned}\eta'_{\text{dragg}}(x) - \eta(x) &= -2\boldsymbol{u}^\mu\,\partial_\mu f(\boldsymbol{\upsilon}^\alpha)d\boldsymbol{\upsilon} - d\tau\xi^\mu_\circ\,\partial_\mu\eta - d\boldsymbol{\upsilon}\boldsymbol{\upsilon}^\mu\,\partial_\mu\eta \\ &= -d\boldsymbol{\upsilon}2\int\limits_{C[p_\circ,p]}\boldsymbol{u}^\nu\nabla_\nu\boldsymbol{\upsilon}^\alpha g_{\alpha\beta}\,dx^\beta - d\tau\xi^\mu_\circ\,\partial_\mu\eta - d\boldsymbol{\upsilon}\boldsymbol{\upsilon}^\mu\,\partial_\mu\eta = 0.\end{aligned}$$

This is equivalent to E with $\delta x = \delta\xi$. But this can be fulfilled if and only if

$$\underset{\xi_\circ}{\bar{\delta}}\,\eta = 0$$

and

$$\underset{\upsilon}{\bar{\delta}}\,\eta = -2\boldsymbol{u}^\mu\,\partial_\mu f(\boldsymbol{\upsilon}) = -2\int\boldsymbol{u}^\nu\nabla_\nu\boldsymbol{\upsilon}^\alpha g_{\alpha\lambda}\,dx^\lambda$$

because the parameters τ and υ are independent. The second equation is nothing else than invariance condition of η for any rigid time-dependent nonrotating motion υ. So, from the invariance of η (equivalently φ) with respect to any transformation ξ (translation or rotation) follows its invariance for any rigid time-dependent nonrotating motion υ. Inserting a motion υ with constant velocity to the last equation one can see that η is constant along symultaneity hyperplane, i.e. space–time must be flat. *Condition* E *is a very strong condition and can be fulfilled in the flat space–time only.*

But one can consider a situation in which η (potential φ) is constant along a given field ξ in any coordinate system, which means that (21) is fulfilled in any reference frame. This imposes a condition on the possible form of the phase f

because transformations rule of η (or φ) depends on f. (21) can be fulfilled in any coordinate system if and only if

$$\boldsymbol{u}^{\mu} \underset{\xi}{\bar{\delta}} \partial_{\mu} f - g^{\mu\nu} \underset{\xi}{\bar{\delta}} \partial_{\mu} f \, \partial_{\nu} f = 0. \tag{22}$$

Assume the space–time is static (i.e., possesses a time-like symmetry ξ). From A, B, and C with $\delta x = \delta\xi$ follows that

$$\xi^{\mu} t_{\mu} = 0 \quad \text{and} \quad \nabla_{\mu}\xi^{\nu} = t_{\mu} g^{\nu\lambda} \, \partial_{\lambda}\varphi.$$

Inserting (20) to (22) one gets

$$\bar{f}(t) = \int\limits_{C[p_{\circ},p]} \boldsymbol{u}^{\rho} \nabla_{\rho} \boldsymbol{v}^{\alpha} g_{\alpha\lambda} \xi^{\lambda} t_{\nu} \, dx^{\nu} \quad \text{and} \quad \nabla_{\mu}\nabla_{\nu}\boldsymbol{v}^{\alpha} = 0.$$

One must be very careful with differentiation of the integral term in (20), because the integral depends on the choice of the integration contour $C[p_{\circ}, p]$. So, if the space–time is static or φ is constant along the integral curves of the time-like field ξ in any reference frame, then

$$\partial_{\mu} f(t) = -g_{\mu\lambda}\boldsymbol{v}^{\lambda} + \frac{1}{2} g_{\alpha\beta}\boldsymbol{v}^{\alpha}\boldsymbol{v}^{\beta} t_{\mu} - t_{\mu} \int\limits_{C[p_{\circ},p]} \boldsymbol{u}^{\rho}\nabla_{\rho}\boldsymbol{v}^{\alpha} g_{\alpha\nu} \, dx^{\nu}$$
$$+ \, t_{\mu} \int\limits_{C[p_{\circ},p]} \boldsymbol{u}^{\rho}\nabla_{\rho}\boldsymbol{v}^{\alpha} g_{\alpha\lambda}\xi^{\lambda} t_{\nu} \, dx^{\nu} + \eta_{\circ} t_{\mu},$$

where $\nabla_{\mu}\nabla_{\nu}\boldsymbol{v}^{\alpha} = \nabla_{\mu}\nabla_{\nu} (\dot{R}^{i}(t)\xi_{i}^{\alpha}) = \dddot{R}^{i}(t)\xi_{i}^{\alpha} t_{\mu} t_{\nu} = 0$, so $\dddot{R}^{i} = 0$ and $\eta_{\circ}$ is an arbitrary constant.

If in addition the space–time is invariant under the action of the rotation group then $\nabla_{\mu}\boldsymbol{v}^{\alpha} = 0$ in the formula given earlier.

5. SCHRÖDINGER EQUATION IN A PRIVILEGED COORDINATE SYSTEM

From A, B, and VI follows that there exist a privileged coordinate system in which

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad (t_{\mu}) = (1, 0, 0, 0).$$

Solving C and D in the coordinate system one gets

$$(\boldsymbol{u}^{\mu}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad (g_{\mu\nu}) = \begin{pmatrix} -2\varphi & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix},$$

where VIII gives

$$\vec{\nabla}^2 \varphi = 0.$$

Inserting this to (14) one gets Schrödinger equation in the privileged frame

$$i\hbar\, \partial_t \Psi = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \Psi + m\varphi\Psi. \tag{23}$$

As an example consider the time-dependent acceleration along x-axis (for simplicity)

$$x \to x + R(t),$$

in the privileged frame. Inserting the transformation to (20) one gets

$$df = -\dot{R}\, dx + \frac{1}{2}\, \dot{R}^2\, dt - \ddot{R} x\, dt + \bar{f}(t)\, dt,$$

and after simple integration

$$f(t, x) = -\dot{R} x + \frac{1}{2} \int_0^t \dot{R}^2\, dt + \bar{f}(t). \tag{24}$$

Formula (24) is well known and was obtained by Kuchař (1980), but in a rather arbitrary way. In the exceptional case when the acceleration is time-independent and $R(t) = \frac{1}{2} a t^2$, where $a = \ddot{R} = constant$

$$f = -atx + \frac{1}{6} a^2 t^3 + \bar{f}(t).$$

If the space–time is static

$$f = -atx + \frac{1}{6} a^2 t^3 + \eta_\circ t + \text{const}. \tag{25}$$

$\eta_\circ$ is an arbitrary constant called *internal energy* that cannot be derived from the invariance conditions even in the flat Galilean space–time, see Lévy-Leblond (1963), or Wawrzycki (N.D.). Formula (25) is also known, see for example Colella *et al*. (1975).

REFERENCES

Colella, R., Overhauser, A. W., and Werner, S. A. (1975). Observation of gravitationally induced quantum interference. *Physical Review Letters* **34**, 1472.

Daŭtcourt, G. (1990). On the Newtonian limit of general relativity. *Acta Physica Polonica B* **21**, 755.

Kasevich, M. and Chu, S. (1991). Atomic interferometry using stimulated Raman transitions. *Physical Review Letters* **67**, 181.

Kuchař, K. (1980). Gravitation, geometry, and nonrelativistic quantum theory. *Physical Review D* **22**, 1285.

Lämmerzahl, C. (1996). On the equivalence principle in quantum theory. *General Relativity and Gravitation* **28**, 1043.

Lévy-Leblond, J. M. (1963). Galilei group and nonrelativistic quantum mechanics. *Journal of Mathematical Physics* **4**, 776.

Penrose, R. (1997). *The Large, the Small and the Human Mind* (chapter 2), Cambridge University Press, Cambridge.

Schouten, J. A. (1951). *Tensor Analysis for Physicists*, Oxford University Press, Oxford. See pages 74–76.

Trautman, A. (1963). Sur la Théorie Newtonienne da la Gravitation. *Comptes Rendus de L'Academie des Sciences de Paris* **247**, 617.

Wawrzycki, J. (2001). Generally Covariant Schrödinger Equation, Part I. *International Journal of Theoretical Physics* **40**(9), 1595.